

# COVARIANT REPRESENTATIONS OF SUBPRODUCT SYSTEMS: INVARIANT SUBSPACES AND CURVATURE

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**ABSTRACT.** Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system of  $C^*$ -correspondences over a  $C^*$ -algebra  $\mathcal{M}$ . Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a pure completely contractive, covariant representation of  $X$  on a Hilbert space  $\mathcal{H}$ , and  $\mathcal{S}$  to be a non-trivial closed subspace of  $\mathcal{H}$ . Then  $\mathcal{S}$  is invariant for  $T$  if and only if there exist a Hilbert space  $\mathcal{D}$ , a representation  $\pi$  of  $\mathcal{M}$  on  $\mathcal{D}$ , and a partial isometry  $\Pi : \mathcal{F}_X \otimes_{\pi} \mathcal{D} \rightarrow \mathcal{H}$  such that

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)\Pi \text{ whenever } \zeta \in X(n), n \in \mathbb{Z}_+, \text{ and}$$

$\mathcal{S}$  is the range of  $\Pi$ , or equivalently,  $P_{\mathcal{S}} = \Pi\Pi^*$ . This result leads us to many important consequences including Beurling type theorem and other general observations on wandering subspaces. We extend the notion of curvature for completely contractive, covariant representations and analyze it in terms of the above results.

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## 1. INTRODUCTION

Arveson in [2] presented an isometric dilation theorem for null (pure) commuting contractions on a given Hilbert space. His result was later extended for pure noncommuting contractions on a Hilbert space by Popescu (cf. [13]) and the isometry providing the dilation in this case is known as the Poisson kernel. Muhly and Solel [11] introduced Poisson kernel for a completely contractive covariant representations over  $W^*$ -correspondences. Arveson considered product system of Hilbert spaces in [1] to explore classification problem for a semigroup of endomorphisms upto cocycle conjugacy. The notion of Poisson kernel for completely contractive covariant representations over a subproduct system of  $W^*$ -correspondences is due to Shalit and Solel [18].

In [16, 17] the first author obtained an invariant subspace theorem for pure contractions and explored invariant subspaces of several analytic reproducing kernel Hilbert spaces. Viselter [19] studied covariant representations on a subproduct system in more details. Covariant representations on a subproduct system provide us a unified approach to study commutative as well as noncommuting contractive tuples on a Hilbert space. Using this unified approach we extend here several results from [16, 17, 4, 14].

We first recall several basic results from [19] including the intertwining property of the Poisson kernel in Section 2. In Section 3 we obtain an invariant subspace theorem

for pure completely contractive representations on standard subproduct systems. As an immediate application we derive a Beurling type theorem.

Section 4 is composed of several results on Wandering subspaces which are motivated from our invariant subspace theorem. This section generalize [4, Section 5] on wandering subspaces for commuting tuple of bounded operators on a Hilbert space.

Our objective in the final section is to extend, several results on curvature of a contractive tuple by Popescu [14, 15], for covariant representations on a subproduct system. We first define the curvature for a covariant representation on a standard subproduct system. This approach is based on the definition of curvature for a covariant representation on a  $W^*$ -correspondence due to Muhly and Solel [9].

## 2. NOTATIONS AND PREREQUISITES

Let  $\mathcal{M}$  be a  $C^*$ -algebra and let  $\mathcal{L}(E)$  be the  $C^*$ -algebra of all adjointable operators of a Hilbert  $\mathcal{M}$ -module  $E$ . The module  $E$  is called a  $C^*$ -correspondence over  $\mathcal{M}$  if it has a left  $\mathcal{M}$ -module structure defined using a non-zero  $*$ -homomorphism  $\phi : \mathcal{M} \rightarrow \mathcal{L}(E)$ , i.e.,  $a\xi := \phi(a)\xi$  for every  $a \in \mathcal{M}$  and  $\xi \in E$ . All such  $*$ -homomorphisms considered in this article are non-degenerate, which means, the closed linear span of  $\phi(\mathcal{M})E$  equals  $E$ . If  $F$  is another  $C^*$ -correspondence over  $\mathcal{M}$ , then we get the notion of tensor product  $F \otimes_{\phi} E$  (cf. [6]) which satisfy the following properties:

$$(\zeta_1 a) \otimes \xi_1 = \zeta_1 \otimes \phi(a)\xi_1,$$

$$\langle \zeta_1 \otimes \zeta_2, \xi_1 \otimes \xi_2 \rangle = \langle \zeta_2, \phi(\langle \zeta_1, \xi_1 \rangle) \xi_2 \rangle$$

for all  $\zeta_1, \zeta_2 \in F$ ,  $\xi_1, \xi_2 \in E$  and  $a \in \mathcal{M}$ .

Assume  $\mathcal{M}$  to be a  $W^*$ -algebra and  $E$  is a Hilbert  $\mathcal{M}$ -module. If  $E$  is self-dual, then  $E$  is called a *Hilbert  $W^*$ -module* over  $\mathcal{M}$ . In this case,  $\mathcal{L}(E)$  becomes a  $W^*$ -algebra (cf. [12]). A  $C^*$ -correspondence over  $\mathcal{M}$  is called a  $W^*$ -correspondence if  $E$  is self-dual, and if the  $*$ -homomorphism  $\phi : \mathcal{M} \rightarrow \mathcal{L}(E)$  is normal. When  $E$  and  $F$  are  $W^*$ -correspondences, then their tensor product  $F \otimes_{\phi} E$  is the self-dual extension of the above tensor product construction.

**Definition 2.1.** Let  $\mathcal{M}$  be a  $C^*$ -algebra,  $\mathcal{H}$  be a Hilbert space, and  $E$  be a  $C^*$ -correspondence over  $\mathcal{M}$ . Assume  $\sigma : \mathcal{M} \rightarrow B(\mathcal{H})$  to be a representation and  $T : E \rightarrow B(\mathcal{H})$  to be a linear map. The tuple  $(T, \sigma)$  is called a covariant representation of  $E$  on  $\mathcal{H}$  if

$$T(a\xi a') = \sigma(a)T(\xi)\sigma(a') \text{ for all } \xi \in E \text{ and } a, a' \in \mathcal{M}.$$

In the  $W^*$ -set up, we additionally assume that  $\sigma$  is normal and that  $T$  is continuous with respect to the  $\sigma$ -topology of  $E$  (cf. [3]) and ultra weak topology on  $B(\mathcal{H})$ . The covariant representation is called completely contractive if  $T$  is completely contractive. The covariant representation  $(T, \sigma)$  is called isometric if

$$T(\xi)^*T(\zeta) = \sigma(\langle \xi, \zeta \rangle) \text{ for each } \xi, \zeta \in E.$$

The following lemma is due to Muhly and Solel [8, Lemma 3.5]:

**Lemma 2.2.** *The map  $(T, \sigma) \mapsto \tilde{T}$  provides a bijection between the collection of all completely contractive, covariant representations  $(T, \sigma)$  of  $E$  on  $\mathcal{H}$  and the collection of all contractive linear maps  $\tilde{T} : E \otimes_{\sigma} \mathcal{H} \rightarrow \mathcal{H}$  defined by*

$$\tilde{T}(\xi \otimes h) := T(\xi)h \text{ for every } \xi \in E \text{ and } h \in \mathcal{H},$$

*and such that  $\tilde{T}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\tilde{T}$  for every  $a \in \mathcal{M}$ . Moreover,  $\tilde{T}$  is isometry if and only if  $(T, \sigma)$  is isometric.*

Assume  $E$  to be a Hilbert space with an orthonormal basis  $\{e_i\}_{i=1}^n$ . Any contractive tuple  $(T_1, T_2, \dots, T_n)$  on a Hilbert space  $\mathcal{H}$  can be realized as a completely contractive, covariant representation  $(T, \sigma)$  of  $E$  on  $\mathcal{H}$  where  $T(e_i) := T_i$  for each  $1 \leq i \leq n$ , and when the representation  $\sigma$  maps every complex number  $\lambda$  to the multiplication operator by  $\lambda$ .

Now we recall several definitions and results from [19] which are essential for our objective. We will use  $A^*$ -algebra, to denote either  $C^*$ -algebra or  $W^*$ -algebra, to avoid repetitions in statements. Similarly we also use  $A^*$ -module and  $A^*$ -correspondence.

**Definition 2.3.** *Assume  $\mathcal{M}$  to be an  $A^*$ -algebra. A collection  $X = (X(n))_{n \in \mathbb{Z}_+}$  of  $A^*$ -correspondences over  $\mathcal{M}$  is called a subproduct system over  $\mathcal{M}$  if  $X(0) = \mathcal{M}$ , and for each  $n, m \in \mathbb{Z}_+$  there exist a coisometric, adjointable bimodule function*

$$U_{n,m} : X(n) \otimes X(m) \rightarrow X(n+m),$$

*such that*

- (a) *the maps  $U_{n,0}$  and  $U_{0,n}$  are the right and the left actions of  $\mathcal{M}$  on  $X(n)$ , respectively; i.e.,*

$$U_{n,0}(\zeta \otimes a) := \zeta a, \quad U_{0,n}(a \otimes \zeta) := a\zeta$$

*for all  $\zeta \in X(n)$ ,  $a \in \mathcal{M}$ ,  $n \in \mathbb{Z}_+$ ;*

- (b) *the following associativity property holds for all  $n, m, l \in \mathbb{Z}_+$ ;*

$$U_{n+m,l}(U_{n,m} \otimes I_{X(l)}) = U_{n,m+l}(I_{X(n)} \otimes U_{m,l}).$$

*If each coisometric maps are unitaries, then we say the family  $X$  is a product system.*

**Definition 2.4.** *Let  $\mathcal{M}$  be an  $A^*$ -algebra and let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a subproduct system over  $\mathcal{M}$ . Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a family of linear transformations  $T_n : X(n) \rightarrow B(\mathcal{H})$ , and define  $\sigma := T_0$ . Then the family  $T$  is called a completely contractive, covariant representation of  $X$  on  $\mathcal{H}$  if*

- (i) *for every  $n \in \mathbb{Z}_+$ , the pair  $(T_n, \sigma)$  is a completely contractive, covariant representation of the  $A^*$ -correspondence  $X(n)$  on  $\mathcal{H}$ ;*

- (ii) *for every  $n, m \in \mathbb{Z}_+$ ,  $\zeta \in X(n)$  and  $\eta \in X(m)$ ,*

$$T_{n+m}(U_{n,m}(\zeta \otimes \eta)) = T_n(\zeta)T_m(\eta). \tag{2.1}$$

For  $n \in \mathbb{Z}_+$  let us define the contractive linear map  $\tilde{T}_n : X(n) \otimes_{\sigma} \mathcal{H} \rightarrow \mathcal{H}$  as

$$\tilde{T}_n(\zeta \otimes h) := T_n(\zeta)h \text{ for all } \zeta \in X(n), h \in \mathcal{H}. \tag{2.2}$$

Thus we can replace Equation 2.1 by

$$\tilde{T}_{n+m}(U_{n,m} \otimes I_{\mathcal{H}}) = \tilde{T}_n(I_{X(n)} \otimes \tilde{T}_m).$$

**Example 2.5.** The Fock space  $\mathcal{F}_X := \bigoplus_{n \in \mathbb{Z}_+} X(n)$  of a subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  is the  $A^*$ -correspondence over  $\mathcal{M}$ . For each  $n \in \mathbb{Z}_+$ , we define a linear map  $S_n^X : X(n) \rightarrow \mathcal{L}(\mathcal{F}_X)$  by

$$S_n^X(\zeta)\eta := U_{n,m}(\zeta \otimes \eta)$$

for every  $m \in \mathbb{Z}_+$ ,  $\zeta \in X(n)$  and  $\eta \in X(m)$ . When  $n \neq 0$  we call each operator  $S_n^X$  a creation operator of  $\mathcal{F}_X$ , and the family  $S^X := (S_n^X)_{n \in \mathbb{Z}_+}$  is called an  $X$ -shift. It is easy to verify that the family  $S^X$  is indeed a completely contractive, covariant representation of  $\mathcal{F}_X$ . From the Definition 2.3 it is easy to see that, for each  $a \in \mathcal{M}$ , the map  $S_0^X(a) = \phi_\infty(a) : \mathcal{F}_X \rightarrow \mathcal{F}_X$  maps  $(b, \zeta_1, \zeta_2, \dots) \mapsto (ab, a\zeta_1, a\zeta_2, \dots)$ .

**Definition 2.6.** Assume  $\mathcal{M}$  to be an  $A^*$ -algebra. A family  $X = (X(n))_{n \in \mathbb{Z}_+}$  of  $A^*$ -correspondences over  $\mathcal{M}$  is called a standard subproduct system if  $X(0) = \mathcal{M}$ , and for any  $n, m \in \mathbb{Z}_+$  the bimodule  $X(n+m)$  is an orthogonally complementable sub-module of  $X(n) \otimes X(m)$ . Let  $E := X(1)$ . Then for each  $n$ , the bi-module  $X(n)$  is an orthogonally complementable sub-module of  $E^{\otimes n}$  (here  $E^{\otimes 0} = \mathcal{M}$ ), and hence we get an orthogonal projection  $p_n \in \mathcal{L}(E^{\otimes n})$  of  $E^{\otimes n}$  onto  $X(n)$ . We denote the orthogonal projection  $\bigoplus_{n \in \mathbb{Z}_+} p_n$  of  $\mathcal{F}_E$ , the Fock space of the product system  $E = (E^{\otimes n})_{n \in \mathbb{Z}_+}$  with trivial unitaries, onto  $\mathcal{F}_X$  by  $P$ .

The projections  $(p_n)_{n \in \mathbb{Z}_+}$  in the previous definition are bimodule maps, and they satisfy

$$p_{n+m} = p_{n+m}(I_{E^{\otimes n}} \otimes p_m) = p_{n+m}(p_n \otimes I_{E^{\otimes m}}) \text{ for all } n, m \in \mathbb{Z}_+.$$

This implies that if we define each  $U_{n,m}$  to be the projection  $p_{n+m}$  restricted to  $X(n) \otimes X(m)$ , then every standard subproduct system becomes a subproduct system over  $\mathcal{M}$ . In this case Equation 2.1 reduces to

$$T_{n+m}(p_{n+m}(\zeta \otimes \eta)) = T_n(\zeta)T_m(\eta) \text{ for all } \zeta \in E^{\otimes n} \text{ and } \eta \in E^{\otimes m},$$

and Equation 2.2 becomes

$$\tilde{T}_{n+m}(p_{n+m} \otimes I_{\mathcal{H}})|_{X(n) \otimes X(m) \otimes_{\sigma} \mathcal{H}} = \tilde{T}_n(I_{X(n)} \otimes \tilde{T}_m). \quad (2.3)$$

Taking adjoints on both the sides we obtain

$$\tilde{T}_{n+m}^* = (I_{X(n)} \otimes \tilde{T}_m^*)\tilde{T}_n^* \text{ for all } n, m \in \mathbb{Z}_+. \quad (2.4)$$

Note that for the sake of convenience we ignored the embedding of  $X(n+m) \otimes_{\sigma} \mathcal{H}$  into  $X(n) \otimes X(m) \otimes_{\sigma} \mathcal{H}$  in the previous formula. We further deduce that

$$\tilde{T}_{n+1}^* = (I_E \otimes \tilde{T}_n^*)\tilde{T}_1^* = (I_{X(n)} \otimes \tilde{T}_1^*)\tilde{T}_n^* \text{ for all } n \in \mathbb{Z}_+, \text{ and} \quad (2.5)$$

$$\tilde{T}_n^* = (I_{X(n-1)} \otimes \tilde{T}_1^*)(I_{X(n-2)} \otimes \tilde{T}_1^*) \dots (I_E \otimes \tilde{T}_1^*)\tilde{T}_1^* \text{ for all } n \in \mathbb{Z}_+. \quad (2.6)$$

**Example 2.7.** If  $X(n)$  is the  $n$ -fold symmetric tensor product of the Hilbert space  $X(1)$ , then  $X = (X(n))_{n \in \mathbb{Z}_+}$  becomes a standard subproduct system of Hilbert spaces (cf. [18, Example 1.3]). Moreover, when  $X(1)$  has orthonormal basis  $\{e_1, e_2, \dots, e_d\}$ , there is a bijection between the set of all completely contractive covariant representations  $T$  of  $X$  on a Hilbert space  $\mathcal{H}$  onto the collection of all commuting row contractions  $(T_1, T_2, \dots, T_d)$  defined on the Hilbert space  $\mathcal{H}$  (cf. [18, Example 5.6]) given by

$$T \leftrightarrow (T(e_1), T(e_2), \dots, T(e_d)).$$

**Remark 2.8.** (i) We use the symbol *sot-lim* for the limit with respect to the strong operator topology. From Equation 2.5 we infer that  $\{\tilde{T}_n \tilde{T}_n^*\}_{n \in \mathbb{Z}_+}$  is a decreasing sequence of positive contractions, and thus  $Q := \text{sot-}\lim_{n \rightarrow \infty} \tilde{T}_n \tilde{T}_n^*$  exists (where *s-lim* stands for limit in the strong operator topology). If  $Q = 0$ , then we say that the covariant representation  $T$  is pure. Note that  $T$  is pure if and only if  $\text{sot-}\lim_{n \rightarrow \infty} \tilde{T}_n^* = 0$ .

(ii) Let  $\psi$  be a representation of  $\mathcal{M}$  on a Hilbert space  $\mathcal{E}$ . Then the induced covariant representation  $S \otimes I_{\mathcal{E}} := (S_n(\cdot) \otimes I_{\mathcal{E}})_{n \in \mathbb{Z}_+}$  is pure, where each  $S_n(\cdot) \otimes I_{\mathcal{E}}$  is an operator from  $X(n)$  into  $B(\mathcal{F}_X \otimes_{\psi} \mathcal{E})$ .

(iii) It is proved in [18, Lemma 6.1] that every subproduct system is isomorphic to a standard subproduct system. Therefore it is enough to consider standard subproduct systems.

**Definition 2.9.** Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$ . We denote the positive operator  $(I_{\mathcal{H}} - \tilde{T}_1 \tilde{T}_1^*)^{1/2} \in B(\mathcal{H})$  by  $\Delta_*(T)$ , and use notation  $\mathcal{D}$  for the defect space  $\overline{\text{Im } \Delta_*(T)}$ .

Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$ . It is proved in [19, Proposition 2.9] that  $\Delta_*(T) \in \sigma(\mathcal{M})'$ . Therefore  $\mathcal{D}$  reduces  $\sigma(a)$  for each  $a \in \mathcal{M}$ . Thus using the reduced representation  $\sigma'$  we can form the tensor product of the Hilbert space  $\mathcal{D}$  with  $X(n)$  for each  $n \in \mathbb{Z}_+$ , and hence with  $\mathcal{F}_X$ . For simplicity we write  $\sigma$  instead of  $\sigma'$ . The *Poisson kernel* of  $T$  is the operator  $K(T) : \mathcal{H} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  defined by

$$K(T)h := \sum_{n \in \mathbb{Z}_+} (I_{X(n)} \otimes \Delta_*(T)) \tilde{T}_n^* h \text{ for all } h \in \mathcal{H}. \quad (2.7)$$

In the next proposition we recall the properties of the Poisson kernel from [19]:

**Proposition 2.10.** Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ . Then  $K(T)$  is a contraction such that

$$K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta) K(T)^* \text{ for all } n \in \mathbb{Z}_+, \zeta \in X(n).$$

Moreover,  $K(T)$  is an isometry if and only if  $T$  is pure.

*Proof.* For each  $h \in \mathcal{H}$ , from Equations 2.5 and 2.3, it follows that

$$\begin{aligned}
\sum_{n \in \mathbb{Z}_+} \|(I_{X(n)} \otimes \Delta_*(T))\tilde{T}_n^* h\|^2 &= \sum_{n \in \mathbb{Z}_+} \langle \tilde{T}_n(I_{X(n)} \otimes \Delta_*(T)^2)\tilde{T}_n^* h, h \rangle \\
&= \sum_{n \in \mathbb{Z}_+} \langle \tilde{T}_n(I_{X(n)} \otimes (I_{\mathcal{H}} - \tilde{T}_1 \tilde{T}_1^*))\tilde{T}_n^* h, h \rangle \\
&= \sum_{n \in \mathbb{Z}_+} \langle \tilde{T}_n \tilde{T}_n^* - \tilde{T}_{n+1} \tilde{T}_{n+1}^* h, h \rangle \\
&= \langle h, h \rangle - \lim_{n \rightarrow \infty} \langle \tilde{T}_n \tilde{T}_n^* h, h \rangle,
\end{aligned}$$

here we also used  $\tilde{T}_0 \tilde{T}_0^* = I_{\mathcal{H}}$ . So  $K(T)$  is a well-defined contraction, and it is an isometry if  $T$  is pure.

For each  $n \in \mathbb{Z}_+$ , let  $z_n \in X(n) \otimes_{\sigma} \mathcal{D}$ . Then we get the adjoint

$$K(T)^* \left( \sum_{n \in \mathbb{Z}_+} z_n \right) = \sum_{n \in \mathbb{Z}_+} \tilde{T}_n(I_{X(n)} \otimes \Delta_*(T))z_n. \quad (2.8)$$

Therefore for every  $m \in \mathbb{Z}_+$ ,  $\eta \in X(m)$  and  $h \in \mathcal{D}$  the Equation 2.5 gives the following relation:

$$\begin{aligned}
K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}})(\eta \otimes h) &= K(T)^*(p_{n+m}(\zeta \otimes \eta) \otimes h) \\
&= \tilde{T}_{n+m}(p_{n+m}(\zeta \otimes \eta) \otimes \Delta_*(T)h) \\
&= \tilde{T}_n(\zeta \otimes \tilde{T}_m(\eta \otimes \Delta_*(T)h)) \\
&= T_n(\zeta)K(T)^*(\eta \otimes h). \quad \square
\end{aligned}$$

### 3. AN INVARIANT SUBSPACE THEOREM

In this section, we define invariant subspace notion for the completely contractive, covariant representations and in the next theorem we obtain a far reaching generalization of [16, Theorem 2.2].

**Definition 3.1.** Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ . A closed subspace  $\mathcal{S}$  of  $\mathcal{H}$  is called invariant for the covariant representation  $T$ , if  $\mathcal{S}$  is invariant for  $\sigma(\mathcal{M})$ , and if  $\mathcal{S}$  is left invariant by each operators in the set  $\{T_n(\zeta) : \zeta \in X(n), n \in \mathbb{N}\}$ .

**Theorem 3.2.** Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a pure completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ , and  $\mathcal{S}$  to be a non-trivial closed subspace of  $\mathcal{H}$ . Then  $\mathcal{S}$  is invariant for  $T$  if and only if there exist a Hilbert space  $\mathcal{D}$ , a representation  $\pi$  of  $\mathcal{M}$  on  $\mathcal{D}$ , and a partial isometry  $\Pi : \mathcal{F}_X \otimes_{\pi} \mathcal{D} \rightarrow \mathcal{H}$  such that  $\mathcal{S}$  is the range of  $\Pi$ , satisfying

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)\Pi \quad \text{whenever } \zeta \in X(n), n \in \mathbb{Z}_+.$$

*Proof.* Since  $\mathcal{S}$  is invariant for  $T = (T_n)_{n \in \mathbb{Z}_+}$ , we get a covariant representation  $(V_n := T_n|_{\mathcal{S}})_{n \in \mathbb{Z}_+}$  of the standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  on  $\mathcal{S}$ . We denote  $V_0$  by  $\pi$ . For each  $n \in \mathbb{N}$ ,  $s \in \mathcal{S}$ , and  $\zeta \in X(n)$ ; the computation

$$\langle \zeta \otimes s, \zeta \otimes s \rangle = \langle s, \pi(\langle \zeta, \zeta \rangle)s \rangle = \langle s, \sigma(\langle \zeta, \zeta \rangle)s \rangle = \langle \zeta \otimes s, \zeta \otimes s \rangle,$$

gives an embedding  $j_n$  from  $X(n) \otimes_\pi \mathcal{S}$  into  $X(n) \otimes_\sigma \mathcal{H}$ . Thus for each  $n \in \mathbb{N}$ ,  $j_n j_n^*$  is an orthogonal projection. Fix  $n \in \mathbb{N}$ , then from the definition of the map  $\tilde{V}_n : X(n) \otimes_\pi \mathcal{S} \rightarrow \mathcal{S}$  we get  $\tilde{V}_n(\zeta \otimes s) = V_n(\zeta)s = T_n(\zeta)s = \tilde{T}_n \circ j_n(\zeta \otimes s)$  for all  $\zeta \in X(n)$  and  $s \in \mathcal{S}$ . It follows that  $\langle \tilde{V}_n \tilde{V}_n^* s, s \rangle = \langle \tilde{T}_n j_n j_n^* \tilde{T}_n^* s, s \rangle \leq \langle \tilde{T}_n \tilde{T}_n^* s, s \rangle$  for every  $n \in \mathbb{N}$  and  $s \in \mathcal{S}$ . Hence the covariant representation  $V$  is pure as well as completely contractive.

We know that the defect space  $\mathcal{D} = \overline{\text{Im } \Delta_*(V)}$  of the representation  $V$  is reducing for  $\pi$ . Thus from the Proposition 2.10, the Poisson kernel  $K(V) : \mathcal{S} \rightarrow \mathcal{F}_X \otimes_\pi \mathcal{D}$  defined by

$$K(V)(s) = \sum_{n \in \mathbb{Z}_+} (I_{X(n)} \otimes \Delta_*(V)) \tilde{V}_n^* s \text{ for all } s \in \mathcal{S}$$

is an isometry, and satisfy  $K(V)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = V_n(\zeta)K(V)^*$  for all  $n \in \mathbb{Z}_+$ , and  $\zeta \in X(n)$ .

Let  $i_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{H}$  be the inclusion map, then clearly  $i_{\mathcal{S}}$  is an isometry and we have  $i_{\mathcal{S}} T_n(\cdot)|_{\mathcal{S}} = T_n(\cdot)i_{\mathcal{S}}$ . Therefore we get a map  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{D} \rightarrow \mathcal{H}$  defined by  $\Pi := i_{\mathcal{S}} K(V)^*$ . Then

$$\Pi \Pi^* = i_{\mathcal{S}} K(V)^* (i_{\mathcal{S}} K(V)^*)^* = i_{\mathcal{S}} i_{\mathcal{S}}^* = P_{\mathcal{S}},$$

the projection on  $\mathcal{S}$ . Hence  $\Pi$  is a partial isometry and the range of  $\Pi$  is  $\mathcal{S}$ . It is also clear from the intertwining property of the Poisson kernel that

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{D}}) = i_{\mathcal{S}} K(V)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = i_{\mathcal{S}} V_n(\zeta) K(V)^* = T_n(\zeta) \Pi, \quad (3.1)$$

since  $i_{\mathcal{S}} V_n = i_{\mathcal{S}} T_n|_{\mathcal{S}} = T_n i_{\mathcal{S}}$ . Conversely, suppose that there exists a partial isometry  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{D} \rightarrow \mathcal{H}$ . Then  $\text{ran } \Pi$  is a closed subspace of  $\mathcal{H}$  and the intertwining relation for  $\Pi$  implies that  $\text{ran } \Pi$  is a  $T = (T_n)_{n \in \mathbb{Z}_+}$  invariant subspace of  $\mathcal{H}$ .  $\square$

**Corollary 3.3.** Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a pure completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ , and  $\mathcal{S}$  to be a non-trivial closed subspace of  $\mathcal{H}$ . Then  $\mathcal{S}$  is invariant for  $T$  if and only if there exist a Hilbert space  $\mathcal{D}$ , a representation  $\pi$  of  $\mathcal{M}$  on  $\mathcal{D}$ , and a bounded linear operator  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{D} \rightarrow \mathcal{H}$  such that  $P_{\mathcal{S}} = \Pi \Pi^*$ , and

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta) \Pi \text{ whenever } \zeta \in X(n), n \in \mathbb{Z}_+.$$

**Definition 3.4.** Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system over an  $A^*$ -algebra  $\mathcal{M}$ . Assume  $\psi$  and  $\pi$  to be representations of  $\mathcal{M}$  on Hilbert spaces  $\mathcal{E}$  and  $\mathcal{E}'$ , respectively. A bounded operator  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{E}' \rightarrow \mathcal{F}_X \otimes_\psi \mathcal{E}$  is called multi-analytic if it satisfies the following condition

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{E}'} ) = (S_n(\zeta) \otimes I_{\mathcal{E}}) \Pi \text{ whenever } \zeta \in X(n), n \in \mathbb{Z}_+.$$

Further we call it inner if it is a partial isometry.

As an application of our invariant subspace theorem, we get the following Beurling type theorem:

**Theorem 3.5.** Assume  $X = (X(n))_{n \in \mathbb{Z}_+}$  to be a standard subproduct system over an  $A^*$ -algebra  $\mathcal{M}$  and assume  $\psi$  to be a representation of  $\mathcal{M}$  on a Hilbert space  $\mathcal{E}$ . Let  $\mathcal{S}$  be a non-trivial closed subspace of the Hilbert space  $\mathcal{F}_X \otimes_\psi \mathcal{E}$ . Then  $\mathcal{S}$  is invariant for  $S \otimes I_{\mathcal{E}}$  if and only if there exist a Hilbert space  $\mathcal{E}'$ , a representation  $\pi$  of  $\mathcal{M}$  on  $\mathcal{E}'$ , and

an inner multi-analytic operator  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{E}' \rightarrow \mathcal{F}_X \otimes_\psi \mathcal{E}$  such that  $\mathcal{S}$  is the range of  $\Pi$ .

*Proof.* Assume  $\mathcal{S}$  is an invariant subspace for  $S \otimes I_{\mathcal{E}}$ . It implies from Theorem 3.2 that we get there exist a Hilbert space  $\mathcal{E}'$ , a representation  $\pi$  of  $\mathcal{M}$  on  $\mathcal{E}'$ , and a partial isometry  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{E}' \rightarrow \mathcal{F}_X \otimes_\psi \mathcal{E}$  such that  $\mathcal{S}$  is the range of  $\Pi$ , and

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{E}'} ) = (S_n(\zeta) \otimes I_{\mathcal{E}}) \Pi \text{ whenever } \zeta \in X(n), n \in \mathbb{Z}_+.$$

For the reverse direction, if we start with a partial isometry  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{E}' \rightarrow \mathcal{F}_X \otimes_\psi \mathcal{E}$ . Then  $\text{ran} \Pi$  is a closed subspace of  $\mathcal{F}_X \otimes_\psi \mathcal{E}$  and the intertwining relation for  $\Pi$  implies that  $\text{ran} \Pi = \mathcal{S}$  is invariant for  $S \otimes I_{\mathcal{E}}$ .  $\square$

#### 4. WANDERING SUBSPACES

We extend the notion of wandering subspaces (cf. [7, p. 561]) for covariant representations of a standard subproduct systems, as follows:

**Definition 4.1.** Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ . A closed  $T$ -invariant subspace  $\mathcal{S}$  of  $\mathcal{H}$  is called wandering for the covariant representation  $T$  if for each  $n = 1, 2, 3, \dots$ ;  $\mathcal{S}$  is orthogonal to  $\mathfrak{L}_n(\mathcal{S}, T) := \bigvee \{T_n(p_n(\zeta))s : \zeta \in E^{\otimes n}, s \in \mathcal{S}\}$ . When there is no confusion we use the notation  $\mathfrak{L}_n(\mathcal{S})$  for  $\mathfrak{L}_n(\mathcal{S}, T)$ , and also use  $\mathfrak{L}(\mathcal{S})$  for  $\mathfrak{L}_1(\mathcal{S})$ . A wandering subspace  $\mathcal{W}$  for  $T$  is called generating if  $\mathcal{H} = \overline{\text{span}}\{\mathfrak{L}_n(\mathcal{W}) : n \in \mathbb{Z}_+\}$ .

**Proposition 4.2.** Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ . If  $\mathcal{S}$  is a closed  $T$ -invariant subspace of  $\mathcal{H}$ , then the subspace  $\mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$  is a wandering subspace for  $T|_{\mathcal{S}} := (T_n|_{\mathcal{S}})_{n \in \mathbb{Z}_+}$ .

*Proof.* For each  $n \in \mathbb{N}$ , and each element  $\eta$  in  $E^{\otimes n}$ ; we get decomposition  $\eta = \xi_1 \otimes \xi_{n-1}$  for some  $\xi_1 \in E, \xi_{n-1} \in E^{\otimes n-1}$ . Fix  $x, s \in \mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$  such that  $y = T_n(p_n(\eta))s \in \mathfrak{L}_n(\mathcal{S} \ominus \mathfrak{L}(\mathcal{S}))$ . Then

$$\begin{aligned} \langle x, y \rangle &= \langle x, T_n(p_n(\eta))s \rangle = \langle x, T_n(p_n(\xi_1 \otimes \xi_{n-1}))s \rangle \\ &= \langle x, T_{1+(n-1)}(p_{1+(n-1)}(\xi_1 \otimes \xi_{n-1}))s \rangle \\ &= \langle x, T_1(\xi_1)T_{n-1}(\xi_{n-1})s \rangle = 0 \end{aligned}$$

since  $\mathcal{S}$  is invariant under  $T_{n-1}(\xi_{n-1})$ . Therefore  $\mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$  is orthogonal to  $\mathfrak{L}_n(\mathcal{S} \ominus \mathfrak{L}(\mathcal{S}))$ , for all  $n \in \mathbb{N}$ , and hence  $\mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$  is a wandering subspace for  $T|_{\mathcal{S}} = (T_n|_{\mathcal{S}})_{n \in \mathbb{Z}_+}$ .  $\square$

Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$ . Suppose  $\mathcal{W}$  is a wandering subspace for  $T$ . Let  $\mathcal{G}_{T, \mathcal{W}} := \bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W})$ . Note that

$$\begin{aligned} \mathfrak{L} \left( \bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W}) \right) &= \overline{\text{span}}\{T_1(p_1(\zeta))T_n(p_n(\eta))w : \zeta \in E, \eta \in E^{\otimes n}, w \in \mathcal{W}, n \in \mathbb{Z}_+\} \\ &= \overline{\text{span}}\{T_{n+1}(p_{n+1}(p_1(\zeta) \otimes p_n(\eta)))w : \zeta \in E, \eta \in E^{\otimes n}, w \in \mathcal{W}, n \in \mathbb{Z}_+\} \\ &\subset \bigvee_{n \in \mathbb{N}} \mathfrak{L}_n(\mathcal{W}). \end{aligned}$$



In the other direction, we have

$$\begin{aligned}
\bigvee_{n \in \mathbb{N}} \mathfrak{L}_n(\mathcal{W}) &= \overline{\text{span}}\{T_n(p_n(p_1(\zeta) \otimes p_{n-1}(\eta)))w : \zeta \in E, \eta \in E^{\otimes n-1}, w \in \mathcal{W}, n \in \mathbb{N}\} \\
&= \overline{\text{span}}\{T_1(p_1(\zeta))T_{n-1}(p_{n-1}(\eta))w : \zeta \in E, \eta \in E^{\otimes n-1}, w \in \mathcal{W}, n \in \mathbb{N}\} \\
&\subset \mathfrak{L}\left(\bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W})\right).
\end{aligned}$$

Thus these sets are equal, and hence

$$\mathcal{G}_{T,\mathcal{W}} \ominus \mathfrak{L}(\mathcal{G}_{T,\mathcal{W}}) = \bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W}) \ominus \mathfrak{L}\left(\bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W})\right) = \mathcal{W}.$$

Hence we have the following proposition:

**Proposition 4.3.** *Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  over an  $A^*$ -algebra  $\mathcal{M}$ . If  $\mathcal{W}$  is a wandering subspace for  $T$ , then*

$$\mathcal{W} = \mathcal{G}_{T,\mathcal{W}} \ominus \mathfrak{L}(\mathcal{G}_{T,\mathcal{W}}).$$

Moreover, if  $\mathcal{W}$  is also generating, then  $\mathcal{W} = \mathcal{H} \ominus \mathfrak{L}(\mathcal{H})$ .

In Theorem 3.2 we observed that each non-trivial closed subspace  $\mathcal{S} \subset \mathcal{H}$ , which is invariant under a pure completely contractive, covariant representation  $T = (T_n)_{n \in \mathbb{Z}_+}$  of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$ , can be written as  $\mathcal{S} = \Pi(\mathcal{F}_X \otimes_\pi \mathcal{D})$ . In the following theorem we study wandering subspaces in a general situation when  $T$  is not necessarily pure.

**Theorem 4.4.** *Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system over an  $A^*$ -algebra  $\mathcal{M}$ . Let  $\pi : \mathcal{M} \rightarrow B(\mathcal{E})$  be a representation on a Hilbert space  $\mathcal{E}$  and  $T = (T_n)_{n \in \mathbb{Z}_+}$  be the covariant representation of  $X$ . Let  $\Pi : \mathcal{F}_X \otimes_\pi \mathcal{E} \rightarrow \mathcal{H}$  be a partial isometry such that  $\Pi(S_n(\zeta) \otimes I_{\mathcal{E}}) = T_n(\zeta)\Pi$  for every  $\zeta \in X(n), n \in \mathbb{Z}_+$ . Then  $\mathcal{S} := \Pi(\mathcal{F}_X \otimes_\pi \mathcal{E})$  is a closed  $T$ -invariant subspace,  $\mathcal{W} := \mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$  is a wandering subspace for  $T|_{\mathcal{S}}$ , and  $\mathcal{W} = \Pi((\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E})$ .*

*Proof.* Define  $F = (\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E}$ . Since  $\mathcal{S}$  is the range of  $\Pi$ , it is a closed  $T$ -invariant subspace. Therefore by the previous proposition, the subspace  $\mathcal{W}$  is a wandering subspace for  $T|_{\mathcal{S}}$ .

$$\begin{aligned}
\mathfrak{L}(\mathcal{S}, T) &= \mathfrak{L}(\Pi(\mathcal{F}_X \otimes_\pi \mathcal{E}), T) \\
&= \bigvee \{T_1(\zeta)k : k \in \Pi(\mathcal{F}_X \otimes_\pi \mathcal{E}), \zeta \in X(1)\} \\
&= \bigvee \{T_1(\zeta)\Pi(l) : l \in \mathcal{F}_X \otimes_\pi \mathcal{E}, \zeta \in X(1)\} \\
&= \bigvee \{\Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e) : l_m \otimes e \in X(m) \otimes_\pi \mathcal{E}, \zeta \in X(1), m \in \mathbb{Z}_+\}.
\end{aligned}$$

For  $x \in (\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E}$  and  $l_m \otimes e \in X(m) \otimes_\pi \mathcal{E}$  we have

$$\begin{aligned}
\langle \Pi x, \Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e) \rangle &= \langle \Pi^* \Pi x, (S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e) \rangle \\
&= \langle x, (S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e) \rangle \\
&= \langle x, P_{1+m}(\zeta \otimes l_m) \otimes e \rangle = 0.
\end{aligned}$$

This computation implies that  $\Pi((\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E}) \subset \mathcal{W}$ . In the converse direction, let us start with  $x \in \mathcal{S} \ominus \mathfrak{L}(\mathcal{S}, T) = \mathcal{W}$ . Since  $x$  is an element of range of the partial isometry  $\Pi$ , there exists an element in  $y \in (\ker \Pi)^\perp$  such that  $\Pi(y) = x$ . Therefore for any  $\zeta \in X(1), \eta \otimes e \in \mathcal{F}_X \otimes_\pi \mathcal{E}$  we have

$$\langle y, (S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e) \rangle = \langle \Pi y, \Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e) \rangle = 0. \quad (4.1)$$

Recall that by definition we have

$$\mathfrak{L}(\mathcal{F}_X \otimes_\pi \mathcal{E}, S \otimes I_{\mathcal{E}}) = \bigvee \{(S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e) : \eta \in X(m), \zeta \in X(1), e \in \mathcal{E}, m \in \mathbb{Z}_+\}.$$

Since  $\mathcal{M} \otimes_\pi \mathcal{E}$  is a generating wandering subspace for the covariant representation  $S \otimes I_{\mathcal{E}}$ , it follows from Proposition 4.3 that  $(\mathcal{F}_X \otimes_\pi \mathcal{E}) \ominus \mathfrak{L}(\mathcal{F}_X \otimes_\pi \mathcal{E}, S \otimes I_{\mathcal{E}}) = \mathcal{M} \otimes_\pi \mathcal{E}$ . Thus Equation 4.1 gives  $y \in \mathcal{M} \otimes_\pi \mathcal{E}$ . Hence  $\Pi((\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E}) = \mathcal{W}$ .  $\square$

**Corollary 4.5.** *With the same notation of Theorem 4.3 we have*

$$\bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W}, T) = \Pi \left( \bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(F, S \otimes I_{\mathcal{E}}) \right)$$

where  $F = (\ker \Pi)^\perp \cap \mathcal{M} \otimes_\pi \mathcal{E}$ . Moreover,  $F$  is wandering subspace for the representation  $S \otimes I_{\mathcal{E}}$ , i.e.,  $F \perp \mathfrak{L}_n(F, S \otimes I_{\mathcal{E}})$  for each  $n \in \mathbb{N}$ .

*Proof.* Let  $f, f' \in F$ , then

$$\begin{aligned} \langle f, (S_n(\zeta) \otimes I_{\mathcal{E}})f' \rangle &= \langle \Pi^* \Pi f, (S_n(\zeta) \otimes I_{\mathcal{E}})f' \rangle = \langle \Pi f, \Pi(S_n(\zeta) \otimes I_{\mathcal{E}})f' \rangle \\ &= 0, \end{aligned}$$

since  $\mathcal{W}$  is wandering subspace for  $T$ . Therefore  $F$  is wandering subspace for the representation  $S \otimes I_{\mathcal{E}}$ .

Since we have  $\mathcal{W} = \Pi F$ ,

$$\begin{aligned} \bigvee \{ \mathfrak{L}_n(\mathcal{W}, T) : n \in \mathbb{Z}_+ \} &= \bigvee \{ (T_n(\zeta) \Pi(F)) : \zeta \in X(n), n \in \mathbb{Z}_+ \} \\ &= \bigvee \{ \Pi(S_n(\zeta) \otimes I_{\mathcal{E}})(F) : \zeta \in X(n), n \in \mathbb{Z}_+ \} \\ &= \Pi \left( \bigvee \{ (S_n(\zeta) \otimes I_{\mathcal{E}})(F) : \zeta \in X(n), n \in \mathbb{Z}_+ \} \right) \\ &= \Pi \left( \bigvee \{ \mathfrak{L}_n(F, S \otimes I_{\mathcal{E}}) : n \in \mathbb{Z}_+ \} \right). \quad \square \end{aligned}$$

## 5. CURVATURE FOR THE COVARIANT REPRESENTATIONS ON SUBPRODUCT SYSTEMS OVER $W^*$ -CORRESPONDENCES

Assume  $X = (X(n))_{n \in \mathbb{Z}_+}$  to be a standard subproduct system of  $W^*$ -correspondences over a semifinite factor  $\mathcal{M}$ . Let  $T = (T_n)_{n \in \mathbb{Z}_+}$  be a completely contractive, covariant representation of  $X$  on a Hilbert space  $\mathcal{H}$ . We define a contractive normal, completely positive map  $\Theta_T : \sigma(\mathcal{M})' \rightarrow \sigma(\mathcal{M})'$  as follows:

$$\Theta_T(a) := \tilde{T}_1(I_E \otimes a) \tilde{T}_1^* \text{ for all } a \in \sigma(\mathcal{M})'.$$

It follows from Equations 2.3, 2.4 and 2.5 that

$$\begin{aligned} \Theta_T^2(a) &= \Theta_T(\Theta_T(a)) = \tilde{T}_1(I_E \otimes (\tilde{T}_1(I_E \otimes a) \tilde{T}_1^*)) \tilde{T}_1^* \\ &= \tilde{T}_1(I_E \otimes \tilde{T}_1)(I_{E^{\otimes 2}} \otimes a)(I_E \otimes \tilde{T}_1^*) \tilde{T}_1^* = \tilde{T}_2(p_2 \otimes I_{\mathcal{H}})(I_{E^{\otimes 2}} \otimes a)(p_2^* \otimes I_{\mathcal{H}}) \tilde{T}_2^* \end{aligned}$$

$$= \tilde{T}_2(I_{X(2)} \otimes a) \tilde{T}_2^* \text{ for all } a \in \sigma(\mathcal{M})'.$$

Inductively, we get

$$\begin{aligned} \Theta_T^n(a) &= \Theta_T(\Theta_T^{n-1}(a)) = \tilde{T}_1(I_E \otimes (\tilde{T}_{n-1}(I_{X(n-1)} \otimes a) \tilde{T}_{n-1}^*)) \tilde{T}_1^* \\ &= \tilde{T}_1(I_E \otimes \tilde{T}_{n-1})(I_E \otimes I_{X(n-1)} \otimes a)(I_E \otimes \tilde{T}_{n-1}^*) \tilde{T}_1^* \\ &= \tilde{T}_n(p_n \otimes I_{\mathcal{H}})(I_E \otimes I_{X(n-1)} \otimes a)(p_n^* \otimes I_{\mathcal{H}}) \tilde{T}_n^* \\ &= \tilde{T}_n(I_{X(n)} \otimes a) \tilde{T}_n^* \text{ for all } a \in \sigma(\mathcal{M})' \text{ and } n \geq 2. \end{aligned}$$

Muhly and Solel [9, Definition 2.5] defined the notion of left dimension (cf. [5]) for a  $W^*$ -correspondences  $E$  over a semifinite factor  $\mathcal{M}$ . We briefly recall this construction as follows:

Assume  $\mathcal{M}$  to be a semifinite factor and  $\tau$  to be a faithful normal semifinite trace. Let  $L^2(\mathcal{M})$  be the GNS construction for  $\tau$ . Then for each  $a \in \mathcal{M}$  we get an operator of left multiplication on  $L^2(\mathcal{M})$  which we denote by  $\lambda(a)$ , and an operator of right multiplication on  $L^2(\mathcal{M})$  we denote by  $\rho(a)$ . Each unital, normal,  $*$ -representation  $\sigma : \mathcal{M} \rightarrow B(\mathcal{H})$  defines a *left  $\mathcal{M}$ -module  $\mathcal{H}$* . There exist an  $\mathcal{M}$ -linear isometry  $V : \mathcal{H} \rightarrow L^2(\mathcal{M}) \otimes l_2$ , here  $\mathcal{M}$ -linear means  $V\sigma(a) = (\lambda(a) \otimes I_{l_2})V$  for all  $a \in \mathcal{M}$ . Note that the projection  $p := VV^* \in (\lambda(\mathcal{M}) \otimes I_{l_2})'$  and  $V\sigma(\mathcal{M})'V^* = p(\lambda(\mathcal{M}) \otimes I_{l_2})'p \subseteq (\lambda(\mathcal{M}) \otimes I_{l_2})'$ . From this relation we can observe that  $(\lambda(\mathcal{M}) \otimes I_{l_2})'$  equals the semifinite factor  $\rho(\mathcal{M}) \otimes B(l_2)$  whose elements can be written as a matrix  $(\rho(a_{ij}))$ . For each positive element  $x \in \sigma(\mathcal{M})'$ , we express  $VxV^*$  in the form  $(\rho(a_{ij}))$ , and we define

$$tr_{\sigma(\mathcal{M})'}(x) := \sum \tau(a_{ii}).$$

Note that  $tr_{\sigma(\mathcal{M})'}$  is a faithful normal semifinite trace on  $\sigma(\mathcal{M})'$ . The *left dimension* of  $\mathcal{H}$  is defined by

$$dim_l(\mathcal{H}) := tr_{\sigma(\mathcal{M})'}(p).$$

For each  $W^*$ -correspondence  $E$ , the Hilbert space  $E \otimes_{\sigma} L^2(\mathcal{M})$  has a natural left  $\mathcal{M}$ -module structure. The left dimension of  $E \otimes_{\sigma} L^2(\mathcal{M})$  will be denoted by  $dim_l(E)$ .

The following proposition is just a reformulation of [9, Proposition 2.12]:

**Proposition 5.1.** *Suppose that  $X = (X(n))_{n \in \mathbb{Z}_+}$  is a standard subproduct system of  $W^*$ -correspondences over a finite factor  $\mathcal{M}$ . Assume that  $E := X(1)$  is a left-finite  $W^*$ -correspondence with  $d := dim_l(E)$ . If  $T = (T_n)_{n \in \mathbb{Z}_+}$  is a completely contractive, covariant representation of  $X$  on  $\mathcal{H}$ , then for every  $x \in \sigma(\mathcal{M})'_+$  we have*

$$tr_{\sigma(\mathcal{M})'}(\Theta_T(x)) \leq \|\tilde{T}_1\|^2 dim_l(E) tr_{\sigma(\mathcal{M})'}(x).$$

**Definition 5.2.** *Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system of  $W^*$ -correspondences over a semifinite factor  $\mathcal{M}$ . Assume that  $E := X(1)$  is a left-finite  $W^*$ -correspondence with  $d := dim_l(E)$ . If  $T = (T_n)_{n \in \mathbb{Z}_+}$  is a completely contractive, covariant representation of  $X$  on a Hilbert space  $\mathcal{H}$ , then we define the curvature of  $T$  by*

$$Curv(T) = \lim_{k \rightarrow \infty} \frac{tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{\sum_{j=0}^{k-1} d^j}.$$

The following result is from [14, p.280].

**Lemma 5.3.** *Assume that  $\{a_j\}_{j=0}^\infty$  and  $\{b_j\}_{j=0}^\infty$  are sequences of  $\mathbb{R}$  such that  $b_j > 0$  for each  $j$ . Consider the partial sums  $A_k := \sum_{j=0}^{k-1} a_j$  and  $B_k := \sum_{j=0}^{k-1} b_j$ , and suppose that  $B_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Then*

$$\lim_{k \rightarrow \infty} \frac{A_k}{B_k} = L,$$

whenever the limit  $L := \lim_{j \rightarrow \infty} \frac{a_j}{b_j}$  exists and it is finite.

**Theorem 5.4.** *Assume  $X = (X(n))_{n \in \mathbb{Z}_+}$  to be a standard subproduct system of  $W^*$ -correspondences over a finite factor  $\mathcal{M}$  such that  $E := X(1)$  is a left-finite  $W^*$ -correspondence with  $d := \dim_l(E)$ . If  $T = (T_n)_{n \in \mathbb{Z}_+}$  is a completely contractive, covariant representation of  $X$  on  $\mathcal{H}$ , then the following holds:*

- (1) *The limit in the definition of  $\text{Curv}(T)$  exists (either as a finite positive number or  $+\infty$ ).*
- (2)  *$\text{Curv}(T) = \infty$  if and only if  $\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T(I)) = \infty$ .*
- (3) *If  $\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T(I))$  is finite, then  $\text{Curv}(T)$  is finite, and in this case we have the following:*
  - (3a) *for  $d \geq 1$ , we get*

$$\text{Curv}(T) = \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(\Theta_T^k(I) - \Theta_T^{k+1}(I))}{d^k},$$

and for  $d > 1$ , we get

$$\text{Curv}(T) = (d - 1) \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{d^k}.$$

(3b) *for  $d < 1$ , the limit  $\lim_{k \rightarrow \infty} \text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))$  is finite and*

$$\text{Curv}(T) = (1 - d) \left( \lim_{k \rightarrow \infty} \text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) \right).$$

*Proof.* Let  $a_k = \Theta_T^k(I) - \Theta_T^{k+1}(I)$  for  $k \geq 0$ . By Proposition 5.1 we have

$$\begin{aligned} a_{k+1} &= \text{tr}_{\sigma(\mathcal{M})'}(\Theta_T(\Theta_T^k(I) - \Theta_T^{k+1}(I))) \\ &\leq \|\tilde{T}_1\|^2 \dim_l(E) \text{tr}_{\sigma(\mathcal{M})'}(\Theta_T^k(I) - \Theta_T^{k+1}(I)) \\ &\leq d a_k \text{ for } k \geq 0. \end{aligned}$$

If  $a_0 = \infty$ , then the fact that  $\{\tilde{T}_n \tilde{T}_n^*\}_{n \in \mathbb{Z}_+}$  is a decreasing sequence of positive contractions implies that  $\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) = \infty$  for all  $k \geq 0$ . If  $a_0 < \infty$ , then  $\{\frac{a_j}{d^j}\}_{j=0}^\infty$  is a non-increasing sequence of non-negative numbers. So its limit exists and we denote it by  $L$ , and we have  $0 \leq L \leq a_0$ .

Let  $d \geq 1$ . Since  $\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) = \sum_{j=0}^{k-1} a_j$ , then by Lemma 5.3 (for  $b_j = d^j$ ) the limit defining  $\text{Curv}(T)$  exists and it is  $L$ . Since  $\sum_{j=0}^{k-1} d^j = \frac{d^k - 1}{d - 1}$  and  $\lim_{k \rightarrow \infty} \frac{d^k - 1}{d^k} = 1$

for  $d > 1$ , we have

$$\begin{aligned} \text{Curv}(T) &= \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{\frac{d^k - 1}{d - 1}} \\ &= (d - 1) \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{d^k - 1} \cdot \lim_{k \rightarrow \infty} \frac{d^k - 1}{d^k} \\ &= (d - 1) \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{d^k}. \end{aligned}$$

This proves statement (3a).

Suppose that  $d < 1$ , then the series  $\sum_{j=0}^{k-1} d^j$  converges to  $1/(1 - d)$ . Since  $a_j \leq d^j a_0$  for all  $j \geq 0$ , the limit  $\lim_{k \rightarrow \infty} \text{tr}_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))$  exists and it is finite. This completes the proof of (3b). The proof of statements (1) and (2) follows by noting that whenever  $a_0$  is finite, the limit defining  $\text{Curv}(T)$  exists and is finite.  $\square$

Recall that  $\Theta_T(x) = \tilde{T}_1(I_E \otimes x)\tilde{T}_1^*$  for all  $x \in \sigma(\mathcal{M})'$ , and that  $Q = \lim_{n \rightarrow \infty} \tilde{T}_n \tilde{T}_n^* = \lim_{n \rightarrow \infty} \Theta_T^n(I_{\mathcal{H}})$ . Using the intertwining property  $K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)K(T)^*$  of the Poisson kernel, we have

$$\begin{aligned} \tilde{T}_n(I_{X(n)} \otimes K(T)^*)(\zeta \otimes k) &= \tilde{T}_n(\zeta \otimes K(T)^*k) \\ &= T_n(\zeta)K(T)^*k \\ &= K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}})k \end{aligned}$$

where  $\zeta \in X(n)$ ,  $k \in \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$ ,  $n \in \mathbb{Z}_+$ . This implies

$$\tilde{T}_n(I_{X(n)} \otimes K(T)^*) = K(T)^*(\widetilde{S_n(\cdot) \otimes I_{\mathcal{D}}}),$$

and hence with help of  $\Theta_T^n(Q) = Q$  and  $K(T)^*K(T) = I_{\mathcal{H}} - Q$  we obtain

$$\begin{aligned} &K(T)^*(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - \Theta_{S \otimes I_{\mathcal{D}}}^n(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}}))K(T) \\ &= K(T)^*K(T) - K(T)^*(S_n(\cdot) \otimes I_{\mathcal{D}})(\widetilde{S_n(\cdot) \otimes I_{\mathcal{D}}})^*K(T) \\ &= K(T)^*K(T) - \tilde{T}_n(I_{X(n)} \otimes K(T)^*)(\widetilde{S_n(\cdot) \otimes I_{\mathcal{D}}})^*K(T) \\ &= I_{\mathcal{H}} - Q - \tilde{T}_n(I_{X(n)} \otimes K(T)^*)(I_{X(n)} \otimes K(T))\tilde{T}_n^* \\ &= I_{\mathcal{H}} - Q - \tilde{T}_n(I_{X(n)} \otimes K(T)^*K(T))\tilde{T}_n^* \\ &= I_{\mathcal{H}} - Q - \tilde{T}_n(I_{X(n)} \otimes (I_{\mathcal{H}} - Q))\tilde{T}_n^* \\ &= I_{\mathcal{H}} - Q - \Theta_T^n(I_{\mathcal{H}} - Q) \\ &= I_{\mathcal{H}} - \Theta_T^n(I_{\mathcal{H}}). \end{aligned}$$

Therefore we have the following proposition:

**Proposition 5.5.** *Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system of  $W^*$ -correspondences over a finite factor  $\mathcal{M}$ . Assume that  $E := X(1)$  is a left-finite  $W^*$ -correspondence with  $d := \dim_l(E)$ . If  $T = (T_n)_{n \in \mathbb{Z}_+}$  is a completely contractive, covariant representation of  $X$  on a Hilbert space  $\mathcal{H}$ , then we can redefine the curvature of  $T$  by*

$$\text{Curv}(T) = \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(K(T)^*(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - \Theta_{S \otimes I_{\mathcal{D}}}^k(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}}))K(T))}{\sum_{j=0}^{k-1} d^j}.$$

**Theorem 5.6.** *Assume  $T = (T_n)_{n \in \mathbb{Z}_+}$  to be a completely contractive, covariant representation of a standard subproduct system  $X = (X(n))_{n \in \mathbb{Z}_+}$  of  $A^*$ -correspondences over*

an  $A^*$ -algebra  $\mathcal{M}$ . Then there exist a Hilbert space  $\mathcal{E}$ , a representation  $\psi$  of  $\mathcal{M}$  on  $\mathcal{E}$ , and an inner multi-analytic operator  $\Pi : \mathcal{F}_X \otimes_{\psi} \mathcal{E} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  such that

$$I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - K(T)K(T)^* = \Pi\Pi^*.$$

*Proof.* Using Proposition 2.10 we obtain the Poisson kernel  $K(T) : \mathcal{H} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  satisfying

$$K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)K(T)^* \text{ for all } n \in \mathbb{Z}_+, \zeta \in X(n).$$

This implies that  $(\text{ran}K(T))^{\perp}$  is invariant for the covariant representation  $S \otimes I_{\mathcal{D}}$ . Now we use Theorem 3.5 and obtain a Hilbert space  $\mathcal{E}$ , a representation  $\psi$  of  $\mathcal{M}$  on  $\mathcal{E}$ , and a partial isometry  $\Pi : \mathcal{F}_X \otimes_{\psi} \mathcal{E} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  such that  $(\text{ran}K(T))^{\perp}$  is the range of  $\Pi$ , and

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{E}}) = (S_n(\zeta) \otimes I_{\mathcal{D}})\Pi \text{ whenever } \zeta \in X(n), n \in \mathbb{Z}_+.$$

Finally, using the fact that  $\Pi$  is a partial isometry and the equation  $(\text{ran}K(T))^{\perp} = \text{ran}(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - K(T)K(T)^*)$ , we get our desired formula.  $\square$

**Proposition 5.7.** *Let  $X = (X(n))_{n \in \mathbb{Z}_+}$  be a standard subproduct system of  $W^*$ -correspondences over a finite factor  $\mathcal{M}$ . Assume that  $E := X(1)$  is a left-finite  $W^*$ -correspondence with  $d := \dim_l(E)$ . If  $T = (T_n)_{n \in \mathbb{Z}_+}$  is a completely contractive, covariant representation of  $X$  on a Hilbert space  $\mathcal{H}$  and  $\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'}(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - \Theta_{S \otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}}))$  is finite, then there exist a Hilbert space  $\mathcal{E}$ , a representation  $\psi$  of  $\mathcal{M}$  on  $\mathcal{E}$ , and an inner multi-analytic operator  $\Pi : \mathcal{F}_X \otimes_{\psi} \mathcal{E} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  such that*

$$\text{Curv}(T) = \lim_{k \rightarrow \infty} \frac{\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - \Pi\Pi^*)(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}} - \Theta_{S \otimes I_{\mathcal{D}}}^k(I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}})))}{\sum_{j=0}^{k-1} d^j}.$$

*Proof.* For simplicity of notation we use  $I$  for  $I_{\mathcal{F}_X \otimes_{\sigma} \mathcal{D}}$  and also use  $\Theta$  for  $\Theta_{S \otimes I_{\mathcal{D}}}$ . Define the representation  $\rho$  of  $\mathcal{M}$  on  $\mathcal{H} \oplus (\mathcal{F}_X \otimes_{\sigma} \mathcal{D})$  by

$$\rho(a) = \begin{pmatrix} \sigma(a) & 0 \\ 0 & \phi_{\infty}(a) \otimes I_{\mathcal{D}} \end{pmatrix} \text{ for all } a \in \mathcal{M}.$$

Therefore we have

$$\begin{aligned} & \text{tr}_{\sigma(\mathcal{M})'}(K(T)^*(I - \Theta^k(I))K(T)) \\ &= \text{tr}_{\rho(\mathcal{M})'} \begin{pmatrix} K(T)^*(I - \Theta^k(I))K(T) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \text{tr}_{\rho(\mathcal{M})'} \left( \begin{pmatrix} 0 & K(T)^*(I - \Theta^k(I))^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (I - \Theta^k(I))^{\frac{1}{2}}K(T) & 0 \end{pmatrix} \right) \\ &= \text{tr}_{\rho(\mathcal{M})'} \left( \begin{pmatrix} 0 & 0 \\ (I - \Theta^k(I))^{\frac{1}{2}}K(T) & 0 \end{pmatrix} \begin{pmatrix} 0 & K(T)^*(I - \Theta^k(I))^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \right) \\ &= \text{tr}_{\rho(\mathcal{M})'} \begin{pmatrix} 0 & 0 \\ 0 & (I - \Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I - \Theta^k(I))^{\frac{1}{2}} \end{pmatrix} \\ &= \text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I - \Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I - \Theta^k(I))^{\frac{1}{2}})}. \end{aligned} \tag{5.1}$$

Theorem 5.6 provides a Hilbert space  $\mathcal{E}$ , a representation  $\psi$  of  $\mathcal{M}$  on  $\mathcal{E}$ , and an inner multi-analytic operator  $\Pi : \mathcal{F}_X \otimes_{\psi} \mathcal{E} \rightarrow \mathcal{F}_X \otimes_{\sigma} \mathcal{D}$  such that

$$\text{Curv}(T) = \lim_{k \rightarrow \infty} \frac{\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I_{\mathcal{F}_X} \otimes_{\sigma} \mathcal{D} - \Pi \Pi^*)(I_{\mathcal{F}_X} \otimes_{\sigma} \mathcal{D} - \Theta_{S \otimes I_{\mathcal{D}}}^k(I_{\mathcal{F}_X} \otimes_{\sigma} \mathcal{D})))}}{\sum_{j=0}^{k-1} d^j}.$$

Using the computation 5.1, Proposition 5.5 we conclude that

$$\begin{aligned} & \text{Curv}(T) \\ &= \lim_{k \rightarrow \infty} \frac{\text{tr}_{\sigma(\mathcal{M})'}(K(T)^*(I - \Theta^k(I))K(T))}{\sum_{j=0}^{k-1} d^j} \\ &= \lim_{k \rightarrow \infty} \frac{\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I - \Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I - \Theta^k(I))^{\frac{1}{2}})}{\sum_{j=0}^{k-1} d^j} \\ &= \lim_{k \rightarrow \infty} \frac{\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I - \Theta^k(I))^{\frac{1}{2}}(I - \Pi \Pi^*)(I - \Theta^k(I))^{\frac{1}{2}})}{\sum_{j=0}^{k-1} d^j} \\ &= \lim_{k \rightarrow \infty} \frac{\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'((I - \Pi \Pi^*)(I - \Theta^k(I)))}{\sum_{j=0}^{k-1} d^j}. \end{aligned}$$

The last equality follows from the following two observations for each  $k$ :

- (1) the recurrence relation  $I - \Theta^k(I) = I - \Theta(I) + \Theta(I - \Theta^{k-1}(I))$  implies that  $\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'(I - \Theta^k(I))} \leq (\sum_{m=0}^{k-1} d^m) \text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'(I - \Theta(I))} < \infty$ ,
- (2) since  $\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'(I - \Theta^k(I))}$  is finite,  $(I - \Theta^k(I))^{\frac{1}{2}}$  belongs to the ideal

$$\{x : \text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'(x^*x)} < \infty\},$$

and hence  $\text{tr}_{(\phi_{\infty}(\mathcal{M}) \otimes I_{\mathcal{D}})'(I - \Theta^k(I))^{\frac{1}{2}}} < \infty$ . □

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